INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS, VOL. 13, 135–144 (1991)

FINITE ELEMENT SOLUTION OF THE INCOMPRESSIBLE NAVIER–STOKES EQUATIONS BY A HELMHOLTZ VELOCITY DECOMPOSITION

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SUMMARY

Finite element solution methods for the incompressible Navier–Stokes equations in primitive variables form are presented. To provide the necessary coupling and enhance stability, a dissipation in the form of a pressure Laplacian is introduced into the continuity equation. The recasting of the problem in terms of pressure and an auxiliary velocity demonstrates how the error introduced by the pressure dissipation can be totally eliminated while retaining its stabilizing properties. The method can also be formally interpreted as a Helmholtz decomposition of the velocity vector.

The governing equations are discretized by a Galerkin weighted residual method and, because of the modification to the continuity equation, equal interpolations for all the unknowns are permitted. Newton linearization is used and at each iteration the linear algebraic system is solved by a direct solver. Convergence of the algorithm is shown to be very rapid. Results are presented for two-dimensional flows in various geometries.

INTRODUCTION

The numerical solution of the incompressible Navier–Stokes equations in primitive variables form presents some particular problems. First, the governing equations, namely continuity and momentum, are in terms of velocity and pressure, with none of the equations identifiable as governing the pressure. Thus various techniques have been developed to overcome this problem. Chorin[1] suggested adding the time derivative of pressure to the continuity equation, thereby identifying it as the pressure one. The added term links the equations and allows pressure to be updated from the continuity equation. The scheme is, however, unsuitable for the time-accurate prediction of unsteady flows. A different scheme has been suggested by Harlow and Welch.[2] Other known algorithms in this context are Patankar and Spalding's[3] SIMPLE and SIMPLER, in which a Poisson pressure or pressure correction equation is solved at every iteration to satisfy

0271-2091/91/120135-10$05.00

Received December 1989
Revised July 1990

the conservation of mass. Other alternatives exist and have been applied in the finite difference and finite element contexts. Taylor and Hughes,[4] for example, solve the equations simultaneously and, to provide stability, use unequal-order elements for pressure and velocity. The consistency of representation of the variables by different degree polynomials in finite elements is known as the Babuška–Brezzi condition.[5,6] It has similarity to the necessity of staggered grids in finite difference solutions[3] to avoid odd–even decoupling of the pressure field, also known as checkerboarding. Recently, a move has been initiated in finite volume methods away from staggered grids and towards the use of collocated solution methods.[7]

Another problem in the solution of the Navier–Stokes equations is the necessity at high Reynolds numbers to use upwinding or directly introduce dissipation in the equations in order to stabilize schemes. This artificial viscosity is introduced arbitrarily and must often be made large, masking the true physical features of the flow.

A third problem is the slow convergence of explicit schemes. It is not unusual for an explicit method to be considered converged when it has reached 'engineering accuracy', a situation often indicating that a limit cycle might be reached if the iteration is continued.

In the present work some approaches are proposed to overcome these problems. First, it is suggested to introduce a pressure Laplacian directly in the continuity equation, thereby interpreting it as a pressure Poisson equation. This term, with a small coefficient proportional to the grid size, can be viewed as an artificial viscosity. It provides the necessary coupling between the equations and circumvents the need for the Babuška–Brezzi condition. An added novelty of this work is to show that, by defining an 'auxiliary' velocity, the scheme can be demonstrated to be free of artificial viscosity, i.e. the scaling coefficient of the pressure Laplacian can be of order unity without affecting the results.

The iteration for the non-linearity is carried out through a Newton linearization, followed by a direct solution of the discretized linear equations at each iteration. The resulting matrices are solved using an efficient vectorized frontal solver adapted and modified from Reference 8. For two-dimensional flows direct solvers are becoming very competitive. Their drawback, however, is the escalation of memory requirements with grid refinement and problem dimension. Advanced matrix technology on vector and parallel computer architectures seems the way to overcome the size limitation for large problems and is the thrust of ongoing work for three-dimensional flows.

In the following the details of the formulation are presented.

FORMULATION OF THE INCOMPRESSIBLE NAVIER–STOKES EQUATIONS

The equations governing steady two-dimensional incompressible viscous flow can be written as

$$\frac{\partial}{\partial x}(u)+\frac{\partial}{\partial y}(v)=0, \tag{1a}$$

$$\frac{\partial}{\partial x}(u^2+p)+\frac{\partial}{\partial y}(uv)-\frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=0, \tag{1b}$$

$$\frac{\partial}{\partial x}(uv)+\frac{\partial}{\partial y}(v^2+p)-\frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)=0. \tag{1c}$$

It is possible to augment the continuity equation with a Laplacian on the right-hand side to provide the necessary coupling and avoid pressure checkerboarding

$$\frac{\partial}{\partial x}(u)+\frac{\partial}{\partial y}(v)=\lambda\left(\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}\right). \tag{2}$$

To eliminate the error in mass continuity associated with an artificial viscosity, the continuity equation is first written as

$$\frac{\partial}{\partial x}(u^*)+\frac{\partial}{\partial y}(v^*)=\lambda\left(\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}\right), \tag{3}$$

with the auxiliary velocity vector $\mathbf{u}^*$ defined as

$$u^*=u+\lambda\frac{\partial p}{\partial x}, \qquad v^*=v+\lambda\frac{\partial p}{\partial y}. \tag{4}$$

The continuity equation remains therefore exact in terms of the physical velocity, i.e. $\nabla\cdot\mathbf{u}=0$. The momentum equations and the boundary conditions can then be rewritten in terms of $\mathbf{u}^*$, the auxiliary velocity. This is straightforward for the convective terms but the viscous terms merit examination. Equation (4) is rewritten in vector form as

$$\mathbf{u}^*=\mathbf{u}+\lambda\nabla p=\mathbf{u}+\hat{\mathbf{u}}. \tag{5}$$

Since $\hat{\mathbf{u}}$ is the gradient of a scalar, $\hat{\mathbf{u}}$ is irrotational; hence

$$\nabla\times\hat{\mathbf{u}}=0. \tag{6}$$

The viscous term of the momentum equations can hence be rewritten as

$$\begin{aligned}\frac{1}{Re}\nabla^2\mathbf{u}&=\frac{1}{Re}(\nabla^2\mathbf{u}^*-\nabla^2\hat{\mathbf{u}})\\&=\frac{1}{Re}[\nabla^2\mathbf{u}^*-\nabla(\nabla\cdot\hat{\mathbf{u}})+\nabla\times(\nabla\times\hat{\mathbf{u}})]\\&=\frac{1}{Re}[\nabla^2\mathbf{u}^*-\nabla(\nabla\cdot\mathbf{u}^*)]\end{aligned} \tag{7a}$$

or

$$\frac{1}{Re}\nabla^2\mathbf{u}=-\frac{1}{Re}\nabla\times(\nabla\times\mathbf{u}^*). \tag{7b}$$

The viscous term can thus be expressed exactly in terms of the auxiliary velocity $\mathbf{u}^*$. The use of equation (7b) may, however, lead to some numerical problems owing to poor conditioning of the resulting matrix. To avoid this, equation (7a) is used with the second term neglected, introducing a small error, proportional to λ, in the viscous term. This error is much less important than the error in the continuity equation of the original scheme, and decreases rapidly with the Reynolds number.

It can be seen that the proposed pressure–velocity explicit coupling or PVEC approach introduces into the continuity equation the stabilizing properities of an artificial viscosity, without the associated error, at the expense of a small error in the viscous terms. It allows equal order of interpolation for pressure and velocity, with no restriction on the choice of elements.[4–6]

The PVEC method can also be interpreted, from equation (5), as a Helmholtz decomposition of the velocity vector into divergence- and curl-free parts. The velocity $\mathbf{u}$ represents the divergence free component of $\mathbf{u}^*$, while the curl-free component $\hat{\mathbf{u}}$ is assumed proportional to the pressure. Thus

$$\nabla\times\mathbf{u}^*=\nabla\times\mathbf{u}-\nabla\times\lambda\nabla p=\nabla\times\mathbf{u}, \tag{8}$$

i.e. the physical velocity vector $\mathbf{u}$ and the auxiliary one $\mathbf{u}^*$ have the same vorticity. In addition, the physical velocity satisfies the incompressibility condition

$$\nabla \cdot \mathbf{u} = 0. \tag{9}$$

One then has the choice of solving the problem in terms of $(\mathbf{u}, p)$, $(\mathbf{u}^*, p)$ or $(\mathbf{u}, \mathbf{u}^*)$. Hafez and co-workers[9,10] develop the $(\mathbf{u}, \mathbf{u}^*)$ system as an alternative to the velocity–vorticity formulation $(\mathbf{u}, \Omega)$.[11,12] In the $(\mathbf{u}, \Omega)$ special care must be taken to ensure mass conservation since the continuity equation is not imposed directly but only through its gradient. The $(\mathbf{u}, \mathbf{u}^*)$ system of equations would consist of solving

$$\mathbf{u}^* = \mathbf{u} + \lambda\left(-\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + \frac{1}{Re}\nabla^2\mathbf{u}\right), \tag{10a}$$

$$\nabla^2\mathbf{u} = -\nabla \times (\nabla \times \mathbf{u}^*). \tag{10b}$$

We prefer the $(\mathbf{u}^*, p)$ formulation because it allows an explicit and direct imposition of the continuity equation. Moreover, in three dimensions it consists of a four-equation system versus six for the $(\mathbf{u}, \mathbf{u}^*)$ system, hence is more amenable to a direct solution.

FINITE ELEMENT DISCRETIZATION

The finite element formulation starts by selecting element interpolation or shape functions for the vector of nodal unknowns $U = (u^*, v^*, p)$:

$$U = \sum_{j=1}^{4} N_j U_j. \tag{11}$$

In this work all variables are interpolated by bilinear shape functions

$$N_j = \tfrac{1}{4}(1 + \xi\xi_j)(1 + \eta\eta_j), \quad j = 1, \ldots, 4, \tag{12}$$

expressed in terms of the normalized non-dimensional parent element co-ordinates ξ_j and η_j.

The Galerkin weighted residual form of the equations can be written as

$$\iint_A W\left\{\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y} - \lambda\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right)\right\} \mathrm{d}A = 0, \tag{13a}$$

$$\iint_A W\left\{\frac{\partial}{\partial x}\left[\left(u^* - \lambda\frac{\partial p}{\partial x}\right)^2 + p\right] + \frac{\partial}{\partial y}\left[\left(u^* - \lambda\frac{\partial p}{\partial x}\right)\left(v^* - \lambda\frac{\partial p}{\partial y}\right)\right] - \frac{1}{Re}\left(\frac{\partial^2 u^*}{\partial x^2} + \frac{\partial^2 u^*}{\partial y^2}\right)\right\} \mathrm{d}A = 0, \tag{13b}$$

$$\iint_A W\left\{\frac{\partial}{\partial x}\left[\left(u^* - \lambda\frac{\partial p}{\partial x}\right)\left(v^* - \lambda\frac{\partial p}{\partial y}\right)\right] + \frac{\partial}{\partial y}\left[\left(v^* - \lambda\frac{\partial p}{\partial y}\right)^2 + p\right] - \frac{1}{Re}\left(\frac{\partial v^*}{\partial x^2} + \frac{\partial^2 v^*}{\partial y^2}\right)\right\} \mathrm{d}A = 0, \tag{13c}$$

where W are weight functions, identical to the interpolation or shape functions N.

After integration by parts, the weak form of the Galerkin system in terms of the auxillary velocity $\mathbf{u}^*$ can be written as

$$\iint_A \left[\left(u^* - \lambda\frac{\partial p}{\partial x}\right)\frac{\partial W}{\partial x} + \left(v^* - \lambda\frac{\partial p}{\partial y}\right)\frac{\partial W}{\partial y}\right] \mathrm{d}A - \oint_S W(\mathbf{u}\cdot\mathbf{n})\, \mathrm{d}S = 0, \tag{14a}$$

$$\iint_A \left\{\left[\left(u^*-\lambda\frac{\partial p}{\partial x}\right)^2+p\right]\frac{\partial W}{\partial x}+\left(u^*-\lambda\frac{\partial p}{\partial x}\right)\left(v^*-\lambda\frac{\partial p}{\partial y}\right)\frac{\partial W}{\partial y}-\frac{1}{Re}\left(\frac{\partial u^*}{\partial x}\frac{\partial W}{\partial x}+\frac{\partial u^*}{\partial y}\frac{\partial W}{\partial y}\right)\right\}\mathrm{d}A$$

$$+\oint_S \frac{1}{Re}W\left(\frac{\partial u^*}{\partial y}n_y-\frac{\partial v^*}{\partial y}n_x\right)\mathrm{d}S-\oint_S W[(\mathbf{u}\cdot\mathbf{n})u+pn_x]\,\mathrm{d}S=0, \tag{14b}$$

$$\iint_A \left\{\left(u^*-\lambda\frac{\partial p}{\partial x}\right)\left(v^*-\lambda\frac{\partial p}{\partial y}\right)\frac{\partial W}{\partial x}+\left[\left(v^*-\lambda\frac{\partial p}{\partial y}\right)^2+p\right]\frac{\partial W}{\partial y}-\frac{1}{Re}\left(\frac{\partial v^*}{\partial x}\frac{\partial W}{\partial x}+\frac{\partial v^*}{\partial y}\frac{\partial W}{\partial y}\right)\right\}\mathrm{d}A$$

$$+\oint_S \frac{1}{Re}W\left(\frac{\partial v^*}{\partial x}n_x-\frac{\partial u^*}{\partial x}n_y\right)\mathrm{d}S-\oint_S W[(\mathbf{u}\cdot\mathbf{n})v+pn_y]\,\mathrm{d}S=0. \tag{14c}$$

NEWTON LINEARIZATION

After substituting the shape and weight functions into equations (14), Newton's method can be introduced by setting

$$U^{n+1}=U^n+\Delta U \tag{15}$$

for the vector of nodal unknowns $U=(u^*, v^*, p)$. After neglecting second-order terms, the continuity and x- and y-momentum equations for incompressible flow, for example, yield respectively

continuity

$$[K_{ij}^{pp}]\{\Delta p\}+[K_{ij}^{pu}]\{\Delta u^*\}+[K_{ij}^{pv}]\{\Delta v^*\}=-\{R_i^p\}, \tag{16a}$$

where the element contributions to the matrices are

$$[k_{ij}^{pp}]=-\lambda\iint_A\left(\frac{\partial W_i}{\partial x}\frac{\partial N_j}{\partial x}+\frac{\partial W_i}{\partial y}\frac{\partial N_j}{\partial y}\right)\mathrm{d}A,$$

$$[k_{ij}^{pu}]=\iint_A\frac{\partial W_i}{\partial x}N_j\,\mathrm{d}A,$$

$$[k_{ij}^{pv}]=\iint_A\frac{\partial W_i}{\partial y}N_j\,\mathrm{d}A$$

and the contribution to the residual is

$$\{r_i^p\}=\iint_A\left[\left(u^*-\lambda\frac{\partial p}{\partial x}\right)\frac{\partial W_i}{\partial x}+\left(v^*-\lambda\frac{\partial p}{\partial y}\right)\frac{\partial W_i}{\partial y}\right]\mathrm{d}A-\oint_S W_i(\mathbf{u}\cdot\mathbf{n})\,\mathrm{d}S;$$

x-momentum

$$[K_{ij}^{up}]\{\Delta p\}+[K_{ij}^{uu}]\{\Delta u^*\}+[K_{ij}^{uv}]\{\Delta v^*\}=-\{R_i^u\}, \tag{16b}$$

where the element contributions to the matrices are

$$[k_{ij}^{up}]=\iint_A\left[\frac{\partial W_i}{\partial x}\left(-2\lambda u_j^*\frac{\partial W_j}{\partial x}+2\lambda^2\frac{\partial p_j}{\partial x}\frac{\partial W_i}{\partial x}+N_j\right)\right.$$
$$\left.+\frac{\partial W_i}{\partial y}\left(-\lambda u_j^*\frac{\partial W_j}{\partial y}-\lambda u_j^*\frac{\partial W_j}{\partial x}+\lambda^2\frac{\partial p_j}{\partial x}\frac{\partial W_j}{\partial y}+\lambda^2\frac{\partial p_j}{\partial y}\frac{\partial W_j}{\partial x}\right)\right]\mathrm{d}A,$$

$$[k_{ij}^{uu}]=\iint_A\left[\frac{\partial W_i}{\partial x}\left(2u_j^*-2\lambda\frac{\partial p_j}{\partial x}\right)N_j+\frac{\partial W_i}{\partial y}\left(v_j^*-\lambda\frac{\partial p_j}{\partial y}\right)N_j-\frac{1}{Re}\left(\frac{\partial W_i}{\partial x}\frac{\partial N_j}{\partial x}+\frac{\partial W_i}{\partial y}\frac{\partial N_j}{\partial y}\right)\right]\mathrm{d}A,$$

$$[k_{ij}^{uv}]=\iint_A\left[\frac{\partial W_i}{\partial y}\left(u_j^*-\lambda\frac{\partial p_j}{\partial x}\right)N_j\mathrm{d}A,\right.$$

$$\{r_i^u\}=\iint_A\left\{\left[\left(u^*-\lambda\frac{\partial p}{\partial x}\right)^2+p\right]\frac{\partial W_i}{\partial x}+\left(u^*-\lambda\frac{\partial p}{\partial x}\right)\left(v^*-\lambda\frac{\partial p}{\partial y}\right)\frac{\partial W_i}{\partial y}\right.$$
$$\left.-\frac{1}{Re}\left(\frac{\partial u^*}{\partial x}\frac{\partial W_i}{\partial x}+\frac{\partial u^*}{\partial x}\frac{\partial W_i}{\partial y}\right)\right\}\mathrm{d}A-\oint_S W_i[(\mathbf{u}\cdot\mathbf{n})u+pn_x]\,\mathrm{d}S+\oint_S\frac{1}{Re}W_i\frac{\partial u^*}{\partial n}\mathrm{d}S;$$

y-momentum

$$[K_{ij}^{up}]\{\Delta p\}+[K_{ij}^{uu}]\{\Delta u^*\}+[K_{ij}^{uv}]\{\Delta v^*\}=-\{R_i^u\},\tag{16c}$$

where the element contributions to the matrices are

$$[k_{ij}^{up}]=\iint_A\left[\frac{\partial W_i}{\partial x}\left(-2\lambda u_j^*\frac{\partial W_j}{\partial x}+2\lambda^2\frac{\partial p_j}{\partial x}\frac{\partial W_i}{\partial x}+N_j\right)\right.$$
$$\left.+\frac{\partial W_i}{\partial y}\left(-\lambda u_j^*\frac{\partial W_j}{\partial y}-\lambda u_j^*\frac{\partial W_j}{\partial x}+\lambda^2\frac{\partial p_j}{\partial x}\frac{\partial W_j}{\partial y}+\lambda^2\frac{\partial p_j}{\partial y}\frac{\partial W_j}{\partial x}\right)\right]\mathrm{d}A,$$

$$[k_{ij}^{uu}]=\iint_A\left[\frac{\partial W_i}{\partial y}\left(v_j^*-\lambda\frac{\partial p_j}{\partial y}\right)N_j\mathrm{d}A,\right.$$

$$[k_{ij}^{uv}]=\iint_A\left[\frac{\partial W_i}{\partial x}\left(u_j^*-\lambda\frac{\partial p_j}{\partial x}\right)N_j+\frac{\partial W_i}{\partial y}\left(2v_j^*-2\lambda\frac{\partial p_j}{\partial y}\right)N_j-\frac{1}{Re}\left(\frac{\partial W_i}{\partial x}\frac{\partial N_j}{\partial x}+\frac{\partial W_i}{\partial y}\frac{\partial N_j}{\partial y}\right)\right]\mathrm{d}A$$

and the contribution to the residual is

$$\{r_i^u\}=\iint_A\left\{\left(u^*-\lambda\frac{\partial p}{\partial x}\right)\left(v^*-\lambda\frac{\partial p}{\partial y}\right)\frac{\partial W_i}{\partial x}+\left[\left(v^*-\lambda\frac{\partial p}{\partial y}\right)^2+p\right]\frac{\partial W_i}{\partial y}\right.$$
$$\left.-\frac{1}{Re}\left(\frac{\partial v^*}{\partial x}\frac{\partial W_i}{\partial x}+\frac{\partial v^*}{\partial y}\frac{\partial W_i}{\partial y}\right)\right\}\mathrm{d}A-\oint_S W_i[(\mathbf{u}\cdot\mathbf{n})v+pn_y]\,\mathrm{d}S+\oint_S\frac{1}{Re}W_i\frac{\partial v^*}{\partial n}\mathrm{d}S.$$

All integrals are evaluated numerically by a 4×4 Gauss–Legendre quadrature after expressing them in local co-ordinates ξ, η. Typically,

$$[k_{ij}]=\int_{-1}^{1}\int_{-1}^{1}k_{ij}[x(\xi,\eta),y(\xi,\eta)]\,|J|\,\mathrm{d}\xi\,\mathrm{d}\eta,\tag{17}$$

where J is the Jacobian of the local transformation at each of the four Gaussian points of an element.

BOUNDARY CONDITIONS

When using the PVEC method there is no longer an explicit appearance of $\mathbf{u}$. Thus equation (5) must be used for boundary conditions where $\mathbf{u}$ must be specified as follows.

For the continuity equation the boundary conditions are:

At inlet The contour integral of equation (14a) is calculated using the specified inlet velocity $\mathbf{u}$.

At exit The pressure is specified as a Dirichlet boundary condition.

On walls The contour integral of equation (14a) drops out naturally because of the no-penetration condition at the wall.

For the momentum equations the boundary conditions are:

At inlet u^*, v^* inlet profiles are imposed as Dirichlet boundary conditions derived from equation (4), i.e. using the imposed physical inlet velocity profile and the pressure gradient from the previous iteration. The inlet values of u^* and v^* thus change from iteration to iteration. The values for u and v, however, remain constant (equal to zero).

At exit $u_n^* = v_n^* = 0$, i.e. the streamlines are parallel. This means that the first contour integral in equations (14b) and (14c) drops out and the second one is calculated at the exit.

On walls $u^* = \lambda p_x$, $v^* = \lambda p_y$, a Dirichlet boundary condition with the pressure obtained from the previous iteration. Similar to the inlet conditions, the values of u^* and v^* on walls change from iteration to iteration. The values for u and v, however, remain constant (equal to zero).

RESULTS

Let us define the $(\mathbf{u}, p)$ approach as being the solution of equations (1b), (1c) and (2) and the $(\mathbf{u}^*, p)$ or PVEC approach as the solution of equations (1b), (1c) and (3). Results are presented for a sudden expansion geometry, at a Reynolds number of 100, using both methods to assess the effect of the pressure dissipation on the continuity equation.

First, tests are carried out using the $(\mathbf{u}, p)$ approach to determine the effect of grid size on the required dissipation λ. The flow is calculated on a fine grid (42 × 19 elements), using the lowest value of λ ($=0{\cdot}005$ for this case) that gives smooth pressure contours, and the results are shown in Figure 1(a). The grid is then coarsened and the lowest value for λ that suppresses wiggles is heuristically determined on each grid. These tests indicate that λ should be proportional to Δ^3, Δ being the mesh size.

Tests are then again run on the finest grid (42 × 19 elements) using both the $(\mathbf{u}, p)$ and $(\mathbf{u}^*, p)$ approaches with a rather high λ ($=0{\cdot}1$). The resulting pressure contours are compared in Figures 1(b) and 1(c) with those of the $(\mathbf{u}, p)$ method with low λ of Figure 1(a). It can be noted that, while both contours are smooth, the $(\mathbf{u}, p)$ method has weaker pressure gradients, reflecting the error in the continuity equation introduced by λ. Moreover, the pressure contours of the $(\mathbf{u}^*, p)$ method at the highest λ ($=0{\cdot}1$), Figure 1(b), compare well with those calculated by the $(\mathbf{u}, p)$ method at the

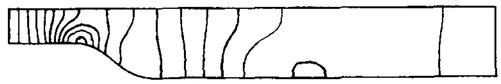

Figure 1(a). Newton–Galerkin incompressible Navier–Stokes algorithm: fine grid, low artificial viscosity coefficient, $\lambda = 0{\cdot}005$, $(\mathbf{u}, p)$ approach

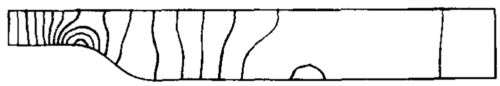

Figure 1(b). Newton–Galerkin incompressible Navier–Stokes algorithm: fine grid, high artificial viscosity coefficient, $\lambda = 0{\cdot}1$, $(\mathbf{u}^*, p)$ or PVEC approach

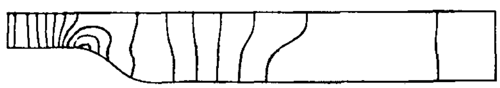

Figure 1(c). Newton–Galerkin incompressible Navier–Stokes algorithm: fine grid, high artificial viscosity coefficient, $\lambda = 0{\cdot}1$, $(\mathbf{u}, p)$ approach

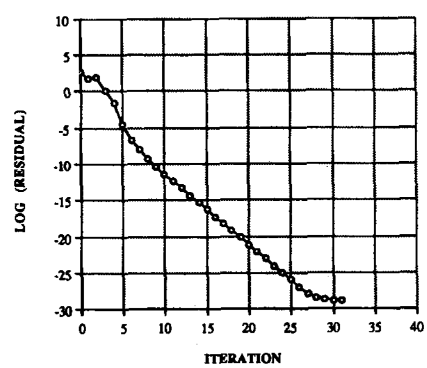

Figure 2. Convergence history of incompressible Navier–Stokes PVEC scheme

lowest λ $(=0{\cdot}005)$, Figure 1(a). This demonstrates that the $(\mathbf{u}^*, p)$ method results are virtually independent of the coefficient λ, even at this relatively low Reynolds number.

Figure 2 shows the convergence history of PVEC scheme for the sudden expansion geometry, demonstrating the rapid convergence of the method. Quadratic convergence is not achieved here because of the lagging of the boundary conditions implementation in the Newton scheme. However, machine accuracy is reached in a few iterations.

The second test case demonstrated is the classical driven cavity problem at a Reynolds number of 400. The problem is run on a 50×50 grid. The centreline velocity profile is plotted in Figure 3 and the streamlines in Figure 4. Both compare very well with the streamfunction–vorticity results of Reference 13 and the results of Reference 14.

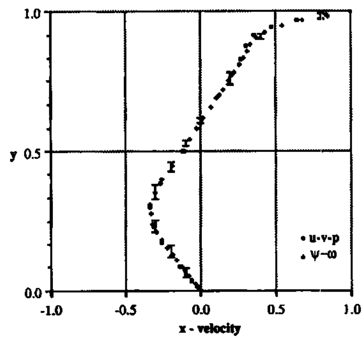

Figure 3. Incompressible viscous flow in a cavity at $Re=400$: centreline velocity

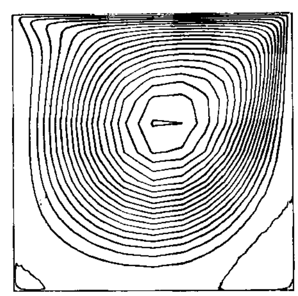

Figure 4. Incompressible viscous flow in a cavity at $Re=400$: streamlines

CONCLUSIONS

This paper presents a stable technique for solving the incompressible Navier–Stokes equations without the need for artificial dissipation. The PVEC method is simple and, more importantly, is more in line with the general spirit of finite element techniques. The method is justified both in a heuristic artificial viscosity way and formally as a Helmholtz decomposition.

The method allows equal interpolation of all variables and removes restrictions on the choice of elements. The implicit technique removes time step size restrictions, and the results underscore the rapid convergence of Newton direct solvers in the solution of the Navier–Stokes equations.

Current work is focusing on extending the methodology to subsonic and transonic flows and on the use of advanced matrix technology as a tool for bringing the three-dimensional situation to manageable levels.

ACKNOWLEDGEMENTS

The collaboration of the engineering staff of Pratt & Whitney Canada is gratefully acknowledged. This work was also partially supported under Operating Grant OGPIN013 and Strategic Grant STREQ040 of the Natural Sciences and Engineering Research Council of Canada (NSERC).

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